

A Symbolic Computation-Based Method and Two Nonlinear Evolution Equations for Water Waves

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A symbolic-computation-based method, which has been newly proposed, is considered for a (2+1)-dimensional generalization of shallow water wave equations and a coupled set of the (2+1)-dimensional integrable dispersive long wave equations. New sets of soliton-like solutions are constructed, along with solitary waves.

1. (2+1)-dimensional Generalized Shallow Water Wave Equation

In order to exactly solve for a (2+1)-dimensional generalization of the shallow water wave equations [1, 2]

$$u_{yt} + u_{xxx} - 3u_{xx}u_y - 3u_x u_{xy} = 0, \quad (1)$$

we introduce a symbolic-computation-based method newly proposed [3, 4], so as to see that

$$u(x, y, t) = A \partial_x w[z(x, y, t)] + \Psi(x, y, t), \quad (2)$$

where A is a non-zero constant, and $\Psi(x, y, t)$, $z(x, y, t)$ and $w(z)$ are differentiable functions.

Then, the requirement is that the result be an ordinary differential equation for $w(z)$, and $z(x, y, t)$ be obtained correspondingly. We substitute (2) back into (1), and equate to zero the coefficients of the terms with the highest power of the differential coefficients of $z(x, y, t)$, i.e., of the $z_x^4 z_y$ terms, so that

$$-6Aw''w''' + w^{(5)} = 0 \rightarrow w(z) = -\frac{2}{A} \ln(z). \quad (3)$$

After a study on the vanishing coefficients of the w' terms, we try

$$\Psi(x, y, t) = \Psi(t), \quad (4)$$

and conjecture that

$$z(x, y, t) = 1 + e^{\gamma x + \Omega(y, t)}, \quad (5)$$

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where $\gamma \neq 0$ is a constant while $\Omega(y, t)$ is a differentiable function.

After symbolic manipulations for the expression of $\Omega(y, t)$, etc., we obtain a set of the soliton-like solutions for (1) as follows:

$$u(x, y, t) = -\gamma \cdot \tanh\left[\frac{\gamma x + \Theta(y) - \gamma^3 t}{2}\right] + \Phi(t), \quad (6)$$

where $\Phi(t) = \Psi(t) - \gamma$ and $\Theta(y)$ are arbitrary differentiable functions. Solitary waves are a special case with $\Theta(y) = ay + b$ and $\Phi(t) = c$, where a , b and c are constants easy to be determined. This solution should be

2. (2+1)-dimensional Integrable Dispersive Long Wave Equations

The coupled set

$$u_{ty} = -\eta_{xx} - \frac{1}{2}(u^2)_{xy}, \quad (7)$$

$$\eta_t = -(u\eta + u + u_{xy})_x \quad (8)$$

has been used to model wide channels or open seas and was studied in [5–7], where $\eta(x, y, t)$ represents the amplitude of a surface wave which propagates in the (x, y) plane with the horizontal velocity $u(x, y, t)$.

Using the aforementioned method, we have done preliminary work aiming at some solutions of (7) and (8) [4]. Recently, we have performed extensive work, which is going to be published elsewhere, with sample solutions (including solitary waves). Here we briefly outline the new families of soliton-like solutions:

Family I: (9)

$$u^{(I)}(x, y, t) = \theta \cdot \left\{ \tanh\left[\frac{\theta \cdot x + \Psi(y, t) - \ln \Sigma(y, t)}{2}\right] + 1 \right\},$$

$$\eta^{(I)}(x, y, t) = \frac{\theta}{2} \cdot \left[\Psi_y(y, t) - \frac{\Sigma_y(y, t)}{\Sigma(y, t)} \right] \cdot \operatorname{sech}^2\left[\frac{\theta \cdot x + \Psi(y, t) - \ln \Sigma(y, t)}{2}\right] - 1, \quad (10)$$

where the differentiable functions $\Sigma(y, t) \neq 0$, $\Psi(y, t)$ and their derivatives, along with the constant $\theta \neq 0$, satisfy

$$\begin{cases} -\theta^2 \Sigma^2 \Psi_y - \Sigma^2 \Psi_t \Psi_y + \Sigma \Sigma_t \Psi_y + \theta^2 \Sigma \Sigma_y + \Sigma \Psi_t \Sigma_y - \Sigma_t \Sigma_y = 0, \\ \Sigma_t \Sigma_y + \Sigma^2 \Psi_{yt} - \Sigma \Sigma_{yt} = 0. \end{cases} \quad (11)$$



Family II:

$$u^{(II)}(x, y, t) = \Theta(y) \cdot \left\{ \tanh \left[\frac{\Theta(y) \cdot x + \Psi(y, t) - \ln \Sigma(y, t)}{2} \right] + 1 \right\}, \quad (12)$$

$$\eta^{(II)}(x, y, t) = \frac{1}{2 \cdot \Sigma(y, t)} \operatorname{sech}^2 \left[\frac{\Theta(y) \cdot x + \Psi(y, t) - \ln \Sigma(y, t)}{2} \right] \cdot [e^{\Theta(y) \cdot x + \Psi(y, t)} \Theta_y(y) + \Sigma(y, t) \Theta_y(y) + x \cdot \Theta(y) \Sigma(y, t) \Theta_y(y) + \Theta(y) \Sigma(y, t) \Psi_y(y, t) - \Theta(y) \Sigma_y(y, t)] - 1, \quad (13)$$

where the differentiable functions $\Theta(y) \neq 0$, $\Sigma(y, t) \neq 0$, $\Psi(y, t)$ and their derivatives satisfy

$$\begin{cases} -\Theta^2 \Sigma - \Sigma \Psi_t + \Sigma_t = 0, \\ \Sigma^2 (\Theta^2)_y + \Sigma_t \Sigma_y + \Sigma^2 \Psi_{yt} - \Sigma \Sigma_{yt} = 0. \end{cases} \quad (14)$$

Family III:

$$u^{(III)}(x, y, t) = \Theta(t) \cdot \left\{ \tanh \left[\frac{\Theta(t) \cdot x - \ln \Phi(t)}{2} \right] + 1 \right\}, \quad (15)$$

$$\eta^{(III)}(x, y, t) = -1, \quad (16)$$

where $\Theta(t) \neq 0$ and $\Phi(t) \neq 0$ are arbitrary, differentiable functions. In this family, $u^{(III)} = u(x, t)$ is independent of y , and $\eta^{(III)}$ is a constant.

Those families give us new information on the motion of the surface waves on open seas and wide channels.

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